Operational Research in Engineering Sciences: Theory and Applications
Vol. 6, Issue 3, 2023, pp. 86-103
ISSN: 2620-1607
eISSN: 2620-1747
DOI: https://doi.org/10.31181/oresta/060304

A STUDY ON VARIABLE QUEUE LENGTH AND SPEED LIMIT ON TOLL PLAZA (ONE-LANE TYPE CASE)

Kent Christian A. Castor[1], Marrick C. Neri[2*]

[1] Division of Physical Sciences and Mathematics, College of Arts and Sciences, University of the Philippines Visayas, Miagao, Iloilo 5023, Philippines
[2] Institute of Mathematics, College of Science, University of the Philippines Diliman, Quezon City 1101, Philippines

Received: 21 May 2023
Accepted: 20 September 2023
First Online: 30 September 2023

Research Paper

Abstract: *The heavy volume of vehicles arriving at toll plazas causes traffic congestion that contributes to longer travel time in toll roads. Utilizing the M/G/1 system under the queuing theory, a nonlinear programming model is developed to identify the appropriate speed limit to be imposed on the freeway of a toll road. The optimal speed limit minimizes the travel time of vehicles in the toll road while it maintains the queue length of vehicles waiting for payment within an arbitrarily set value. A formula solving for the optimal speed limit of a toll plaza operating a one type of tollbooth is developed by simplifying the constraints and observing the behavior of the objective function. The formula is tested using available traffic data.*

Keywords: *Nonlinear Programming, Queuing Theory, Optimization, Toll Road System, Travel Time, Queue Length*

[*]Corresponding author: mcneri@up.edu.ph (M. C. Neri)
kacastor@up.edu.ph (K. C. A. Castor)

1. Introduction

The toll road or expressway is a high-speed highway for which payment is necessary for usage. The road is designed for nonstop travel resulting in less travel time. As more and more vehicles ply the highways, an alternative to avoid traffic congestions is to utilize the convenience of fast travel along toll roads. However, there are build-ups of lines of vehicles in exit points as motorists slow down to stop to pay the toll fee. Even with the imposition of the radio-frequency identification (RFID) on most toll plazas, there are still vehicles lining up to pay cash. The result are long queues which lead to motorist dissatisfaction and potential losses in revenues.

Studies on reducing traffic congestion in toll roads date back to the 1950s. In the pioneering work of Edie (1954), he investigated the optimal number of toll collectors to be utilized that maximized the quality of service and minimized the expenses. In the works of Nuredini and Ramadani (2011), Busam (2005) and Kim (2009), the natural behavior of traffic around pay-toll plazas were illustrated using queuing theory. Nuredini and Ramadini determined to identify the optimal number of open tollbooths that minimizes the waiting time of the vehicles with the assumption that the toll lanes follow a M/M/1 queuing system. They attributed delays in toll plazas to wasted time in tollbooths. Their results were obtained by an exhaustive method which computed the travel time in all possible number of tollbooths.

Both the works of Busam and Kim assumed that the toll lanes follow a M/G/1 queuing system. In Kim (2009), the optimal lane configuration for the mix of methods of payment is determined so that the waiting time in queue at the toll plaza is minimized. Busam developed a non-linear integer program with constraints on the total arrival rates, number of toll booths available, and the relation of the arrival and service rates. On the other hand, Kim presented a decision-making model for designing a new toll plaza where the optimal lane configuration minimizes both the toll company's cost and the driver's cost which is based on a given value of the waiting time. An integer programming model was developed minimizing the total cost of operation with the number of each type of tollbooth as the decision variable. Constraints on the arrival rates, total number of tollbooths available, bounds, relations of the arrival rate, and service rate were considered.

The work of Van Dijk et al. (1999) combined the queuing theory and simulation study in a hybrid approach to identify an optimal tollbooth configuration. Structural designs with specifications on features such as spacing, safety, and accessibility were considered for the optimal configuration. Performance indicators like waiting times, queue lengths, and workload of a toll booth were taken into account.

There were also studies on factors such as speed and the number of lanes to improve the efficiency of toll road operations. Yang et al. (2013) proposed a variable speed limit control system along a freeway to improve the capacity of a downstream bottleneck. The goal of the study is to minimize the total travel time over a certain section in a toll road subject to limitation in the mean speed, density boundary and transition flow. They observed that with the obtained optimal speed limit, travel time was effectively reduced.

The work of Kesten, Göksu, and Akbaş (2013) investigated the outcome of a variable speed limit to control freeway recurring traffic congestion. The authors developed variable speed limit strategies for which speed limit and road occupancy are fixed. O'Dea (1999), using a modified bottle congestion model, focused on metering toll road entrance to address the congestion problem. Here, the rate at which vehicles can enter the road is controlled.

In Mittal and Sharma (2022), the study proposed a simulation model depicting the driver's pattern in selecting lanes and identifying the most appropriate lane and payment type for the toll plaza's minimum delay. The simulation results are based more on the driver's response than on speed policies implemented on the tollway.

Another strategy to reduce queueing time in a toll plaza is by utilizing reversible roadways, i.e., one or two lanes in the other off-peak direction are borrowed to decrease congestion in the peak direction (Wolshon & Lambert, 2004). Kumar, Thakare, and Tawalare (2020) presented results of the reversible lane concept for collection of tolls by optimizing the toll servers according to the demand from a particular side, resulting in optimization of queue length/queue time. Although this concept is widely used, it is outside the scope of this paper.

Increasing the speed limit decreases the time spent by vehicles in the freeway. However, as a consequence, the arrival rate of vehicles in the queue area for payment increases. In effect, it increases the queue length of vehicles for payment and time spent in the toll plaza is increased. Longer travel times spent in the toll road and queue length is interpreted as a poor performance of a toll road system.

This study is conducted with the aim of refining existing investigations on the performance of toll roads. The direct effect of speed limit to the arrival rate of vehicles as a factor affecting the travel time in the toll road is explored. The queue length of vehicles waiting for payment is considered as an essential part in the performance of the toll road. Other factors affecting delays in the toll plaza such as the service time in tollbooths are considered part of the travel time across the toll road. Instead of the commonly used M/M/1 system, the M/G/1 queuing system is used, that is, no assumption on the nature of the service time is made. Analysis on the mathematical model leads to the derivation of a formula that associates the speed limit with the variable queue length.

The following list are notations used in this paper:

d	is the length of the freeway,
S	is the current speed limit,
n	is the number of tollbooths open,
m	is the arbitrarily set maximum queue length (arbitrary queue length),
D	is the traffic density,
μ	is the mean service rate of one tollbooth,
σ	is the standard deviation of the service time of one tollbooth,
λ	is the current mean arrival rate of the toll plaza,
s	is the variable speed limit,
s^*	is the optimal speed limit,
$\bar{\lambda}$	is the mean arrival rate of the toll plaza associated with s
$\bar{\rho}$	is the traffic intensity,
$\bar{l}$	is the length of vehicles in one tollbooth,
$\bar{l}_Q$	is the queue length of vehicles in one tollbooth,
T	is the mean travel time of vehicles in the toll road,
$T_{freeway}$	is the mean travel time of vehicles to cover the freeway, and
T_{plaza}	is the mean time spent by of vehicles in the toll plaza

In the next section, the mathematical optimization model is derived. The third section provides a formula that gives the solution to the model. The paper ends with some numerical examples.

2. Materials

Travel time on a toll road can be divided into two parts: time travelled in the freeway and time spent at the toll plaza. The time spent on the freeway is the time to cover the distance from the toll road entrance to the toll plaza queue area while the time spent at the toll plaza is the time spent by vehicles waiting in queue to pay in cash the toll fee.

Speed limit is the maximum speed at which vehicles can travel along a certain road as allowed by law. In the Philippines, the current speed limit S is imposed by the Toll Regulatory Board (TRB), and in many toll roads, the value of S is either 80 kph or 100 kph. The main purpose of a speed limit is to reduce traffic incidents and to improve road safety. In toll roads, the speed limit has a direct effect on the flow of vehicles. By controlling the speed limit, the arrival of vehicles in the toll plaza can be managed, and consequently, the travel time in the toll road can be lessened.

The goal is to construct a model that identifies the optimal speed limit on a toll road while maintaining the queue length of vehicles waiting to pay cash within an arbitrarily set value. The objective is to minimize travel time subject to toll road regulations and queue constraints. Some assumptions on the structure of the toll road are considered in the formulation of the model, as follows:

A1. Vehicles have only one entrance to the toll road and the only exit for vehicles out of the toll road is the toll plaza where the toll dues are collected.
A2. The flow of vehicles within the toll road is uninterrupted, that is, there are no factors causing delays within the freeway.
A3. All vehicles in the freeway travel at the speed limit.
A4. The density of vehicles in the toll road is constant.
The structure of the toll road can be observed in Figure 1.

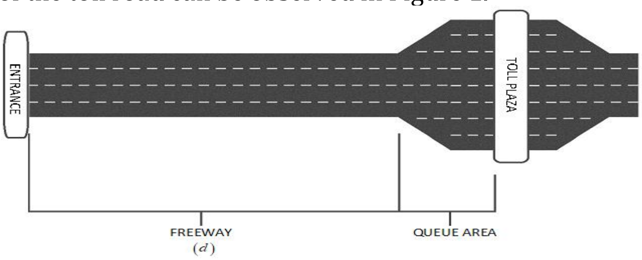

Figure 1: Toll Road Structure for One Lane-type Case

2.1. Speed Limit and Arrival Rate Relation

Allowing vehicles to run up to the speed limit S could result in long payment queues at the toll plaza. On the other, imposing another speed limit s could place the length of queues at a value more to the motorists' satisfaction. Considering that traveling at a speed greater than the current limit is a traffic violation, a natural constraint is

$$s \leq S \tag{1}$$

The relation between the speed limit and the arrival rate is established by the Flow-Density-Speed Formula (Roess, Prassas, & McShane, 2004),

$$Flow = Density \cdot Speed.$$

With the assumption that the flow within the freeway is uninterrupted, the average flow of vehicle across the freeway would be constant, thereby making the flow rate of vehicles equal to the arrival rate of vehicles in the queue area. With the current speed limit S and current arrival rate λ, we have

$$\lambda = D \cdot S.$$

Holding the density constant, the new arrival rate $\overline{\lambda}$ associated with the speed limit s is

$$\overline{\lambda} = D \cdot s,$$

or equivalently,

$$\overline{\lambda} = \frac{\lambda}{S} \cdot s\,. \tag{2}$$

It is important to note that the Flow-Density-Speed formula used in establishing the relation between the speed limit and arrival rate is applicable only in practical ideal conditions. This relationship is corrected in the real traffic flow by different factors such as the width of the traffic lanes, the distance of lateral obstacles and number of access points, among others.

2.2. Arrival and Service Process

The arrival rate and volume of vehicles in the toll plaza are random, therefore the arrival process is a Poisson process. With the assumption that drivers would try to find and join the lane with the shortest queue length, the mean arrival rate per lane can be obtained by dividing the mean arrival rate to the entire plaza by the number of open tollbooths. Thus, with n tollbooths open, and $\overline{\lambda}$ the optimal arrival rate, the mean arrival rate to one tollbooth is

$$\frac{\overline{\lambda}}{n}.$$

Using equation (2), the mean arrival rate to one tollbooth can be expressed as

$$\frac{\lambda s}{nS}. \tag{3}$$

According to Kim (2009), it is not realistic to assume that the service time follows an exponential distribution. The use of exponential distribution for the service time would overestimate the queue length and waiting time. In this study the service time is assumed to have a general distribution with mean service rate μ and standard deviation σ.

2.2.1. Traffic Intensity

With the mean arrival rate to one tollbooth obtained in (3) and with mean service rate μ at a tollbooth, the traffic intensity at one tollbooth, denoted by $\overline{\rho}$, is

$$\overline{\rho} = \frac{\lambda s}{Sn\mu}. \tag{4}$$

For stability of the system, the traffic intensity must be less than one ($\rho < 1$). Therefore, another constraint for the speed limit is considered, that is,

$$s < \frac{Sn\mu}{\lambda}. \tag{5}$$

2.2.2. System Length

With the traffic intensity ρ, the mean service rate μ and the standard deviation of the service time σ, the system length is obtained by the *Pollaczek-Khinchin Equation*. With the traffic intensity obtained in (4), the mean length of vehicles at one tollbooth, denoted by $\bar{l}$, is

$$\bar{l} = \frac{\lambda s}{Sn\mu} + \frac{s^2\lambda^2(1+\sigma^2\mu^2)}{2Sn\mu(Sn\mu - \lambda s)}. \tag{6}$$

Thus, the mean queue length at one tollbooth, denoted by $\bar{l}_Q$, is

$$\bar{l}_Q(s) = \frac{s^2\lambda^2(1+\sigma^2\mu^2)}{2Sn\mu(Sn\mu - \lambda s)}. \tag{7}$$

It is assumed that most drivers, regardless of how fast the service is, still evaluate the toll plaza poorly because of the queue length. To maintain the queue length within an arbitrarily set maximum value m, the following constraint needs to be considered:

$$\bar{l}_Q(s) \leq m. \tag{8}$$

2.3. Travel Time

Upon entering the toll road, the vehicles would be in a freeway where the speed limit s is to be imposed. The average travel time of vehicles to cover the freeway is [10]

$$T_{freeway} = \frac{d}{s}. \tag{9}$$

The travel time of vehicles in the toll plaza is the time spent by vehicles in queue for payment and the transaction time in the toll booth. We implement the M/G/1 queuing system to determine the average time spent by vehicles in the toll plaza.

Using the system length and the arrival rate, the time spent by the vehicles in the plaza is obtained by *Little's Law*. With the mean arrival rate and mean length of vehicles obtained in (3) and (6) respectively, the mean time spent by a vehicle in the toll plaza, denoted by T_{plaza}, is

$$T_{plaza} = \frac{1}{\mu} + \frac{s\lambda(1+\sigma^2\mu^2)}{2\mu(Sn\mu - \lambda s)}. \tag{10}$$

The travel time of vehicle in the entire toll road is the sum of the times spent on the freeway and on the toll plaza. Denoted by T, the mean travel time of vehicle in the toll road is

$$T = T_{freeway} + T_{plaza},$$

or equivalently,

$$T = \frac{d}{s} + \frac{1}{\mu} + \frac{s\lambda(1+\sigma^2\mu^2)}{2\mu(Sn\mu - \lambda s)}. \quad (11)$$

3. Nonlinear Model

The goal is to minimize the mean travel time in the toll road subject to the speed and length considerations. Specifically, the following univariate model is proposed:

$$(P_0) \quad \text{Minimize} \quad T(s) = \frac{d}{s} + \frac{1}{\mu} + \frac{s\lambda(1+\sigma^2\mu^2)}{2\mu(Sn\mu - \lambda s)} \quad (12)$$

$$\text{Subject to} \quad s \le S \quad (13)$$

$$s < \frac{Sn\mu}{\lambda} \quad (14)$$

$$\frac{s^2\lambda^2(1+\sigma^2\mu^2)}{2Sn\mu(Sn\mu - \lambda s)} \le m \quad (15)$$

$$s > 0 \quad (16)$$

The restriction on the optimal speed limit as stated in law is presented in the first constraint (13). The stability of the traffic intensity is considered in the second constraint (14). The third constraint (15) maintains the queue length within an arbitrarily set value m. Later, m may be recommended satisfiability length or the length of the vehicles for a better performance of the toll plaza. Lastly, in (16), the optimal speed limit may only take a positive value.

The nonlinear program in its present form may be ill-conditioned due to presence of strict inequality constraints. The model will be modified and since the objective function is convex, as will be seen later, the existence of a solution will be shown.

3.1. Simplified Model

The system constraints may be simplified by removing possibly redundant constraints.

3.1.1. Second Constraint

With the current speed limit S, the current arrival rate to one tollbooth is

$$\frac{\lambda S}{nS} = \frac{\lambda}{n}.$$

Thus, it follows that the traffic intensity of one lane is $\frac{\lambda}{n\mu}$, and for stability, we require

$$\lambda < n\mu. \quad (17)$$

From the first constraint, $s \le S$, we conclude from (2) that

$$\overline{\lambda} \le \lambda. \quad (18))$$

Incorporating (17) and (18), we get

$\bar{\lambda} \leq n\mu.$

From the value of $\bar{\lambda}$ in (2), we obtain the second constraint

$$s < \frac{Sn\mu}{\lambda},$$

thereby making it redundant.

3.1.2. Combining Third and Fourth Constraints

Consider the length constraint

$$\frac{s^2\lambda^2(1+\sigma^2\mu^2)}{2Sn\mu(Sn\mu-\lambda s)} \leq m.$$

Let $C = \frac{\lambda^2(1+\sigma^2\mu^2)}{2Sn\mu}$. The third constraint can be written as

$$\frac{s^2C}{Sn\mu-\lambda s} - m \leq 0.$$

Solving for the value of s by completing the square, the constraint on the queue length can be expressed as

$$-\sqrt{\frac{mSn\mu}{C} + \left(\frac{m\lambda}{2C}\right)^2} - \frac{m\lambda}{2C} \leq s \leq \sqrt{\frac{mSn\mu}{C} + \left(\frac{m\lambda}{2C}\right)^2} - \frac{m\lambda}{2C}. \tag{19}$$

Note that the left-hand side of (19) is negative. Since $s > 0$, we get

$$0 < s \leq \bar{s}. \tag{20}$$

where

$$\bar{s} = \sqrt{\frac{mSn\mu}{C} + \left(\frac{m\lambda}{2C}\right)^2} - \frac{m\lambda}{2C}.$$

Consequently, constraints (13), (16), and (20) can be merged to the constraint

$$0 < s \leq \min(\bar{s}, S). \tag{21}$$

Incorporating all the constraints, the model can be simplified to

$$(P)\begin{cases} \text{Minimize} \quad T(s) = \dfrac{d}{s} + \dfrac{1}{\mu} + \dfrac{s\lambda(1+\sigma^2\mu^2)}{2\mu(Sn\mu-\lambda s)} \\ \text{Subject to} \qquad\qquad 0 < s \leq \hat{s} \end{cases}$$

where $\hat{s} = \min(\bar{s}, S)$.

Let I be the interval $\left(0, \frac{Sn\mu}{\lambda}\right)$. From (14) and (16), a solution s of (P) is in I. Clearly, $\bar{s}$ is also in this interval. The next lemma shows that the queue length $\bar{l}_Q(s)$ is increasing over I. Hereon, assumptions A1-A4 hold.

Lemma 3.1. *The queue length $\bar{l}_Q(s)$ is increasing over I.*

Proof. The critical values of $\bar{l}_Q(s)$ are 0, $\frac{Sn\mu}{\lambda}$, and $\frac{2Sn\mu}{\lambda}$. Sign analysis shows that the derivative $\bar{l}'_Q(s)$ is positive in $\left(0, \frac{Sn\mu}{\lambda}\right)$.

The following observations are evident:

- The queue length at the speed limit $s = \overline{s}$ is exactly m, i.e., $\overline{l}_Q(s) = m$.
- If the queue length at $s = S$ is equal to m, then $S = \overline{s}$.

4. Solutions of the Model

In this section, we derive a general solution for the nonlinear model (P). The process makes use of the convexity if $T(s)$ and elementary optimality conditions.

Theorem 4.1. *The objective function T is convex over the interval I.*

Proof. In the function $T(s) = \frac{d}{s} + \frac{1}{\mu} + \frac{s\lambda(1+\sigma^2\mu^2)}{2\mu(Sn\mu-\lambda s)}$, let

$$f_1(s) = \frac{d}{s}, \text{ and}$$
$$f_2(s) = \frac{1}{u} + \frac{s\lambda(1+\sigma^2\mu^2)}{2\mu(Sn\mu-\lambda s)}.$$

Since the second derivative f_1'' is positive over positive values, f_1 is convex over the interval $(0, +\infty)$. On the other hand, the second derivative of f_2 is positive over the interval $\left(-\infty, \frac{Sn\mu}{\lambda}\right)$. Thus, f_2 is convex thereon. This implies $T(s) = f_1(s) + f_2(s)$ is convex, even strictly, on the intersection

$$(0, +\infty) \cap \left(-\infty, \frac{Sn\mu}{\lambda}\right) = \left(0, \frac{Sn\mu}{\lambda}\right).$$

Under stability conditions of traffic intensity, there is $0 < S < \frac{Sn\mu}{\lambda}$. Consequently, $0 < \min(\overline{s}, S) < \frac{Sn\mu}{\lambda}$. This yields the next corollary.

Corollary 4.1. *The* objective *function T is convex over the interval* $(0, \min(\overline{s}, S))$.

Prior to obtaining a solution to the model, we first consider the minimizer of T on I as given by the following lemma.

Lemma 4.1. *The function T has a* local *minimizer on I. This local minimizer is*

$$\ddot{s} = \begin{cases} \dfrac{Sn\mu}{d\lambda^2 - Sn\mu K}\left(\lambda d - \sqrt{Sn\mu dK}\right) & if\ Sn\mu K - d\lambda^2 \neq 0 \\ \dfrac{Sn\mu}{2\lambda} & otheswise \end{cases}$$

where $K = \frac{\lambda(1+\sigma^2\mu^2)}{2\mu}$.

Proof. Since T is strictly convex on I, a stationary point of T in I is a unique minimizer. The derivative of T is

$$T' = -\frac{d}{s^2} + \frac{Sn\mu K}{(Sn\mu - \lambda s)^2}$$
$$= \frac{(Sn\mu K - d\lambda^2)s^2 + 2Sn\mu d\lambda s - d(Sn\mu)^2}{s^2(Sn\mu - \lambda s)^2}.$$

There are two possible cases.

Case 1. Suppose $Sn\mu K - d\lambda^2 = 0$

The critical numbers of T are $0, \frac{Sn\mu}{\lambda}, \ddot{s}$, where $\ddot{s} = \frac{Sn\mu}{2\lambda}$. Clearly, $0 < \ddot{s} < \frac{Sn\mu}{\lambda}$, and the first derivative test gives $\ddot{s}$ as the minimizer in I.

Case 2. Suppose $Sn\mu K - d\lambda^2 \neq 0$

The critical numbers of T are $0, \frac{Sn\mu}{\lambda}, s_1$, and s_2, where

$$s_1 = \frac{Sn\mu\sqrt{d}}{\lambda\sqrt{d} + \sqrt{Sn\mu K}} \text{ and } s_2 = \frac{Sn\mu\sqrt{d}}{\lambda\sqrt{d} - \sqrt{Sn\mu K}}.$$

Clearly, $s_1 \in I$.

- Let $\lambda\sqrt{d} - \sqrt{Sn\mu K} < 0$. Then $s_2 < 0$. Also $Sn\mu K - d\lambda^2 > 0$ and

$$T'(s) = \frac{(s - s_1)(s - s_2)}{s^2(Sn\mu - \lambda s)^2}.$$

Applying the first derivative test, s_1 is a local minimizer in I.

- Let $\lambda\sqrt{d} - \sqrt{Sn\mu K} > 0$. Then we have the ordering

$$0 < s_1 < \frac{Sn\mu}{\lambda} < s_2 .$$

Since $Sn\mu K - d\lambda^2 < 0$, we have

$$T'(s) = \frac{-(s - s_1)(s - s_2)}{s^2(Sn\mu - \lambda s)^2}.$$

Applying the first derivative test again, s_1 is a local minimizer in I.

Regularizing s_1 yields $\ddot{s}$. In all cases, T has a local minimizer in I.

The next theorem gives us a formula for obtaining the solution of problem (P).

Theorem 4.2. *Let $\ddot{s}$ be defined as in Lemma 1.1. The optimal solution of (P) is*

$$s^* = \min(\ddot{s}, \bar{s}, S). \quad (22)$$

Proof. We consider the following cases.

Case 1: Let $\bar{s} \leq S$. By the constraint of (P), the solution space is $(0, \bar{s}\,]$.

Suppose $\ddot{s}$ is in $(0, \bar{s}\,]$. Since the solution space is a subset of I and $\ddot{s}$ is a local minimizer of T, it follows that it is also the optimal solution s^* of the nonlinear model. Moreover, since $\ddot{s} \leq \bar{s} \leq S$, the optimal solution s^* is the minimum of $\ddot{s}, \bar{s}$ and S, that is,

$$s^* = \min(\ddot{s}, \bar{s}, S).$$

Suppose $\ddot{s}$ is not in the solution space. Then $\bar{s} \leq \ddot{s}$ and $(0, \bar{s}\,]$ is a subset of the interval $(0, \ddot{s})$ where T is decreasing. Consequently, T is strictly decreasing over the solution space. Therefore, the optimal solution s^* of the nonlinear model is the upperbound of the solution space, which is $\bar{s}$. Moreover $\bar{s} \leq \ddot{s}$ and $\bar{s} \leq S$, the optimal solution s^* is the minimum of $\ddot{s}, \bar{s}$ and S, that is,

$$s^* = \min(\ddot{s}, \bar{s}, S).$$

Case 2: Let $S \leq \bar{s}$. The solution space is $(0, S\,]$.

Suppose $\ddot{s}$ is in the solution space. Since $\ddot{s}$ is a local minimizer of T, it follows that it is also the optimal solution s^* of the nonlinear model. Moreover, since $\ddot{s} \leq \bar{s} \leq S$

the optimal solution is

$s^* = \min(\ddot{s}, \overline{s}, S)$.

Suppose $\ddot{s}$ is not in the solution space. It implies that $S < \ddot{s}$ and the solution space $(0, S\,]$ is a subset of $(0, \ddot{s})$ where T is decreasing. Consequently, T is strictly decreasing over the solution space. Therefore, the optimal solution of the nonlinear model is at the upperbound of the solution space, which is S. Moreover, since $S \leq \ddot{s}$ and $S \leq \overline{s}$, the optimal solution is

$s^* = \min(\ddot{s}, \overline{s}, S)$.

For all the different cases, the result shows that the optimal solution is always the minimum of $\ddot{s}$, $\overline{s}$ and S.

Theorem 4.2 provides a formula for the optimal solution of (P). This formula avoids the computational cost of running a solution algorithm, while guaranteeing the exact solution. In the next section, numerical experiments are presented that validate the correctness of the formula.

5. Numerical Result

5.1. The Data

The data in Kim (2009) is used to check the validity of the model developed. Kim designed a case study to test his model in the location of a potential toll plaza. He obtained the values he used from a toll plaza that operated the same lane-types. Each lane-type is considered a separate case, thereby assuming that the toll plaza operates only a one lane-type.

The three basic modes of payment are Manual, Automatic Coin Machine (ACM) and Electronic Toll Collection (ETC). Manual mode of payment employs toll collectors to gather payment. Automatic Coin Machine (ACM) uses coin machines where motorists drop the payment. Lastly, the Electronic Toll Collection (ETC) utilizes transponders and external sensors that collect tolls even before a vehicle reaches the gate. In the data, the mean service rates of Manual, ACM, and ETC lane-types were $\mu_M = 6.1$, $\mu_A = 10.2$, and $\mu_E = 49.8$ vehicles per minute, respectively. The standard deviation of the service times for the lane-types were $\sigma_M = 0.12$, $\sigma_A = 0.06$, and $\sigma_E = 0.01$ minute per vehicle, respectively. Table 1 shows the hourly mean arrival rate of vehicles to different lane-types. The number of open lane-types is based on the minimum number of required open lanes on an hourly basis. Table 2 shows the minimum number of lanes to open for each lane-type on an hourly basis.

The fixed distance from the entrance to the queue area or the freeway and the current speed limit being imposed are not available in Kim (2009) since its main goal is to identify the optimal configuration. For our test, we assume the speed limit from where the author was. In Seoul, Korea, the speed limit on expressways is generally 100 km per hour (1.66667 km per minute). The fixed distance of the freeway is assumed to be 5 km. We set the upper bound m for the queue length to be 5 vehicles.

Table 1. Mean Arrival Rates (vehicles per minute)

Time	Manual λ_M	ACM λ_A	ETC λ_E
24:00-1:00	9.1	4.2	3.1
1:00-2:00	9.3	4.3	3.5
2:00-3:00	8.9	4.5	3.3
3:00-4:00	12.3	5.2	4.1
4:00-5:00	15.7	4.9	5.7
5:00-6:00	20.4	7.3	7.9
6:00-7:00	32.4	17.5	21.6
7:00-8:00	45.2	25.1	30.4
8:00-9:00	36.7	20.8	24.2
9:00-10:00	27	18.2	18.2
10:00-11:00	24.3	15.3	14.8
11:00-12:00	23.2	13.5	12.4
12:00-13:00	22.7	12.6	11.5
13:00-14:00	23.5	13.1	12
14:00-15:00	24.2	14.4	13.2
15:00-16:00	26.3	16.3	15.4
16:00-17:00	28	17.7	17.2
17:00-18:00	31.7	18.5	19.7
18:00-19:00	36.8	20.6	24.1
19:00-20:00	34.9	18.9	23.3
20:00-21:00	30.4	14.8	20.3
21:00-22:00	20.1	9.2	10.9
22:00-23:00	15.6	6.3	6.6
23:00-24:00	10.4	5	3.9

Table 2. Minimum Required Number of Lanes to Open

Time	Manual n_M	ACM n_A	ETC n_E
24:00-1:00	2	1	1
1:00-2:00	2	1	1
2:00-3:00	2	1	1
3:00-4:00	3	1	1
4:00-5:00	3	1	1
5:00-6:00	4	1	1
6:00-7:00	6	2	1
7:00-8:00	8	3	1
8:00-9:00	7	3	1
9:00-10:00	5	2	1
10:00-11:00	4	2	1
11:00-12:00	4	2	1
12:00-13:00	4	2	1
13:00-14:00	4	2	1
14:00-15:00	4	2	1
15:00-16:00	5	2	1
16:00-17:00	5	2	1
17:00-18:00	6	2	1
18:00-19:00	7	3	1
19:00-20:00	6	2	1
20:00-21:00	5	2	1
21:00-22:00	4	1	1
22:00-23:00	3	1	1
23:00-24:00	2	1	1

5.2. Numerical Solutions: Analytic and Algorithmic

Theorem 4.2 is used to solve for the optimal speed limit and its corresponding travel time and queue length of vehicles. Table 3 shows the optimal values for the manual lane-type.

Table 3. Optimal Values for the Manual Lane-type using Theorem 4.2

Time	Speed Limit (km/min)	Queue Length (veh/min)	Travel Time (min/veh)
24:00-1:00	1.666666667	1.681406471	3.53347431
1:00-2:00	1.666666667	1.877238804	3.567641696
2:00-3:00	1.666666667	1.510837296	3.503448425
3:00-4:00	1.666666667	1.058082026	3.422003213
4:00-5:00	1.590844533	2.843368228	3.876133299
5:00-6:00	1.628604554	2.800326556	3.795963689
6:00-7:00	1.546099375	2.896448522	3.976086276
7:00-8:00	1.483622913	2.974636986	4.125503088
8:00-9:00	1.588216084	2.846422713	3.881852545
9:00-10:00	1.546099375	2.896448522	3.976086276
10:00-11:00	1.388508541	3.103885906	4.378201769
11:00-12:00	1.448562153	3.020764677	4.214872672
12:00-13:00	1.477658952	2.98236489	4.140412272
13:00-14:00	1.431661434	3.043616308	4.259481817
14:00-15:00	1.393756253	3.096404291	4.363380295
15:00-16:00	1.583507292	2.851914188	3.892144822
16:00-17:00	1.49570033	2.959132419	4.09566731
17:00-18:00	1.577144	2.859375131	3.906148887
18:00-19:00	1.584290098	2.85099953	3.890429653
19:00-20:00	1.444771147	3.025854875	4.224790111
20:00-21:00	1.387463858	3.105380473	4.38116544
21:00-22:00	1.650665824	2.775882178	3.750784362
22:00-23:00	1.600114922	2.832656808	3.856108496
23:00-24:00	1.600114922	2.832656808	3.856108496

For instance, in the time interval 24:00-1:00, the optimal speed limit is 1.666666667 km/min which is the same with the current speed limit. In this interval the optimal speed limit generates a queue length of 1.681406471 vehicles per minute and a travel time of 3.53347431 minutes per vehicle. For the time interval 4:00-5:00, the optimal speed limit is 1.590844533 km/min which generates a queue length of 2.843368228 vehicles per minute and a travel time of 3.876133299 minutes per vehicle.

The optimal values were also obtained using a numerical method on model (P). The optimal solution is obtained by the Golden search algorithm over the region identified by simplifying the constraints involved. The Golden search is applicable in this case since the objective can be shown to be convex over the restricted solution space.

For a toll plaza operating with a manual lane-type only, Table 4 shows the comparison of the values obtained using the two methods. On the average, the absolute error of approximation between the optimal speed limit is 2.13E-06 and 8.45E-07 for the optimal travel time in the toll road. Values were also computed for the ACM and ETC lane-types. The average absolute approximation errors between the optimal speed limits and travel times are 4.25E-06 and 5.67E-06 respectively for the ACM lane-type. For the ETC lane -type, 4.97E-06 and 8.9E-06 respectively are the average absolute approximation errors between optimal speed limits and travel times. With an error of 10^{-6}, the values obtained by Theorem 4.2 and model (P) are numerically the same. These results, we are able to show numerically that the formula developed in Theorem 4.2 is accurate in finding the optimal speed limit.

Table 4. Comparison of Two Methods for Manual Lane-type

Time	Speed Limit			Travel Time in Toll Road		
	With Thm 4.2	With (P)	Error	With Thm 4.2	With (P)	Error
24:00-1:00	1.6667	1.6667	4.97E-06	3.5335	3.5335	4.61E-06
1:00-2:00	1.6667	1.6667	4.97E-06	3.5676	3.5676	3.88E-06
2:00-3:00	1.6667	1.6667	4.97E-06	3.5034	3.5035	5.20E-06
3:00-4:00	1.6667	1.6667	4.97E-06	3.422	3.422	6.60E-06
4:00-5:00	1.5908	1.5908	1.47E-06	3.8761	3.8761	1.49E-11
5:00-6:00	1.6286	1.6286	1.06E-07	3.796	3.796	7.02E-14
6:00-7:00	1.5461	1.5461	1.61E-06	3.9761	3.9761	1.95E-11
7:00-8:00	1.4836	1.4836	2.51E-06	4.1255	4.1255	5.48E-11
8:00-9:00	1.5882	1.5882	1.72E-06	3.8819	3.8819	2.03E-11
9:00-10:00	1.5461	1.5461	1.61E-06	3.9761	3.9761	1.95E-11
10:00-11:00	1.3885	1.3885	1.63E-06	4.3782	4.3782	2.92E-11
11:00-12:00	1.4486	1.4486	2.36E-06	4.2149	4.2149	5.26E-11
12:00-13:00	1.4777	1.4777	1.51E-06	4.1404	4.1404	2.02E-11
13:00-14:00	1.4317	1.4317	2.70E-06	4.2595	4.2595	7.21E-11
14:00-15:00	1.3938	1.3938	9.68E-07	4.3634	4.3634	1.02E-11
15:00-16:00	1.5835	1.5835	1.20E-06	3.8921	3.8921	1.00E-11
16:00-17:00	1.4957	1.4957	6.47E-07	4.0957	4.0957	3.55E-12
17:00-18:00	1.5771	1.5771	1.29E-06	3.9061	3.9061	1.17E-11
18:00-19:00	1.5843	1.5843	4.25E-07	3.8904	3.8904	1.26E-12
19:00-20:00	1.4448	1.4448	1.80E-06	4.2248	4.2248	3.10E-11
20:00-21:00	1.3875	1.3875	1.44E-06	4.3812	4.3812	2.29E-11
21:00-22:00	1.6507	1.6507	1.05E-06	3.7508	3.7508	6.69E-12
22:00-23:00	1.6001	1.6001	2.56E-06	3.8561	3.8561	4.40E-11
23:00-24:00	1.6001	1.6001	2.56E-06	3.8561	3.8561	4.40E-11

5.2. Speed Limit: Current and Optimal Values

Table 5. Current and Optimal Values for Manual Lane-type

Time	Current Values			Optimal Values		
	Speed Limit (km/min)	Queue Length (veh/min	Travel Time (min/veh)	Speed Limit (km/min)	Queue Length (veh/min)	Travel Time (min/veh)
24:00-1:00	1.6667	1.6814	3.5335	1.6667	1.6814	3.5335
1:00-2:00	1.6667	1.8772	3.5676	1.6667	1.8772	3.5676
2:00-3:00	1.6667	1.5108	3.5034	1.6667	1.5108	3.5034
3:00-4:00	1.6667	1.0581	3.422	1.6667	1.0581	3.422
4:00-5:00	1.6667	3.9782	3.9241	1.5908	2.8434	3.8761
5:00-6:00	1.6667	3.2743	3.806	1.6286	2.8003	3.796
6:00-7:00	1.6667	5.2441	4.1351	1.5461	2.8964	3.9761
7:00-8:00	1.6667	8.9303	4.7445	1.4836	2.9746	4.1255
8:00-9:00	1.6667	4.0371	3.9339	1.5882	2.8464	3.8819
9:00-10:00	1.6667	5.2441	4.1351	1.5461	2.8964	3.9761
10:00-11:00	1.6667	185.8379	33.7545	1.3885	3.1039	4.3782
11:00-12:00	1.6667	14.1162	5.5978	1.4486	3.0208	4.2149
12:00-13:00	1.6667	9.5395	4.8449	1.4777	2.9824	4.1404
13:00-14:00	1.6667	19.3114	6.451	1.4317	3.0436	4.2595
14:00-15:00	1.6667	92.1557	18.3963	1.3938	3.0964	4.3634
15:00-16:00	1.6667	4.1464	3.9522	1.5835	2.8519	3.8921
16:00-17:00	1.6667	7.8956	4.5739	1.4957	2.9591	4.0957
17:00-18:00	1.6667	4.3028	3.9783	1.5771	2.8594	3.9061
18:00-19:00	1.6667	4.1279	3.9491	1.5843	2.851	3.8904
19:00-20:00	1.6667	15.0325	5.7483	1.4448	3.0259	4.2248
20:00-21:00	1.6667	232.6799	41.4336	1.3875	3.1054	4.3812
21:00-22:00	1.6667	2.957	3.7524	1.6507	2.7759	3.7508
22:00-23:00	1.6667	3.7822	3.8913	1.6001	2.8327	3.8561
23:00-24:00	1.6667	3.7822	3.8913	1.6001	2.8327	3.8561

Using the current speed limit, the corresponding queue length and travel time for each time period is computed.

On an hourly basis, Table 5 shows the current and optimal speed limit with their corresponding queue lengths and travel time for the manual lane-type. These values are graphically shown in Figures 2 and 3.

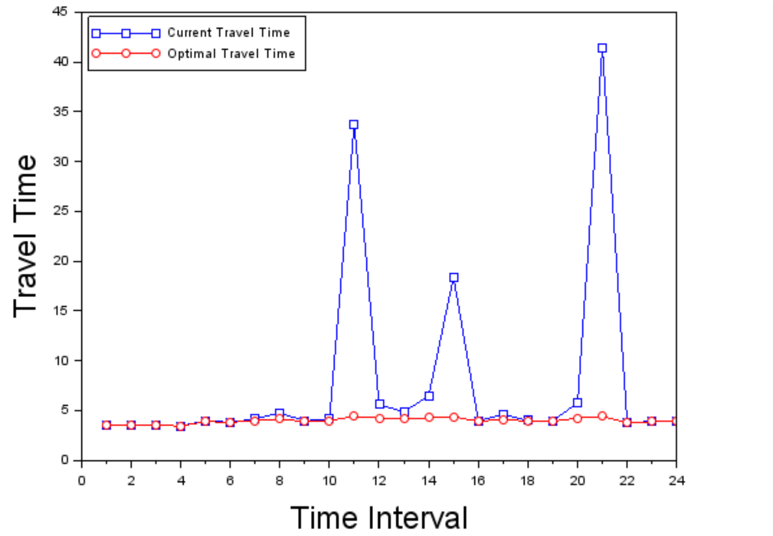

Figure 2: Travel Time: Current vs Optimal for Manual Lane-type

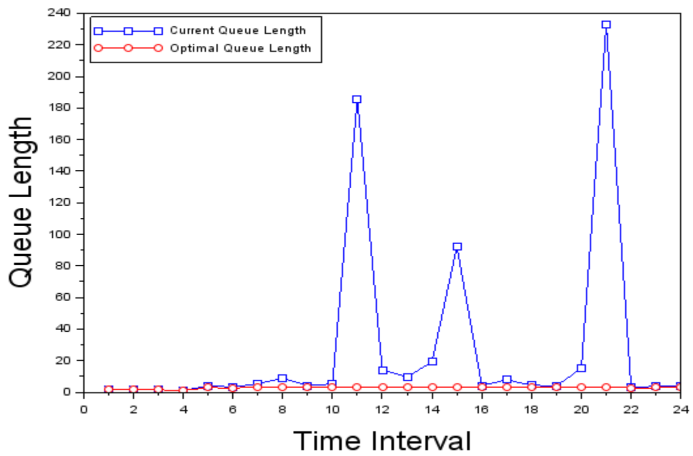

Figure 3: Queue Length: Current vs Optimal for Manual Lane-type

Figure 2 and 3 shows that the queue length and travel time are observed to be large at time intervals 10:00-11:00, 14:00-15:00 and 20:00-21:00. These are attributed to the mean arrival rates of the vehicles and the number of lanes serving. Also, the queue lengths for time intervals 6:00-7:00, 7:00-8:00, 9:00-10:00, 11:00-12:00, 12:00-13:00, 13:00-14:00, 14:00-15:00, 16:00-17:00, 19:00-20:00, 20:00-21:00 are all greater than 5. This observation implies with (P) that the current speed limit S is excluded from the solution space.

Table 6. Comparison of Current and Optimal Values for the Manual Lane-type

Time	Travel Time			Queue Length		
	Current	Optimal	Decrease	Current	Optimal	Decrease
24:00-1:00	3.5335	3.5335	0	1.6814	1.6814	0
1:00-2:00	3.5676	3.5676	0	1.8772	1.8772	0
2:00-3:00	3.5034	3.5034	0	1.5108	1.5108	0
3:00-4:00	3.422	3.422	0	1.0581	1.0581	0
4:00-5:00	3.9241	3.8761	0.048	3.9782	2.8434	1.1348
5:00-6:00	3.806	3.796	0.01	3.2743	2.8003	0.474
6:00-7:00	4.1351	3.9761	0.159	5.2441	2.8964	2.3477
7:00-8:00	4.7445	4.1255	0.619	8.9303	2.9746	5.9557
8:00-9:00	3.9339	3.8819	0.0521	4.0371	2.8464	1.1906
9:00-10:00	4.1351	3.9761	0.159	5.2441	2.8964	2.3477
10:00-11:00	33.7545	4.3782	29.3763	185.8379	3.1039	182.734
11:00-12:00	5.5978	4.2149	1.3829	14.1162	3.0208	11.0954
12:00-13:00	4.8449	4.1404	0.7045	9.5395	2.9824	6.5571
13:00-14:00	6.451	4.2595	2.1915	19.3114	3.0436	16.2678
14:00-15:00	18.3963	4.3634	14.0329	92.1557	3.0964	89.0593
15:00-16:00	3.9522	3.8921	0.0601	4.1464	2.8519	1.2945
16:00-17:00	4.5739	4.0957	0.4782	7.8956	2.9591	4.9365
17:00-18:00	3.9783	3.9061	0.0722	4.3028	2.8594	1.4434
18:00-19:00	3.9491	3.8904	0.0587	4.1279	2.851	1.2769
19:00-20:00	5.7483	4.2248	1.5235	15.0325	3.0259	12.0067
20:00-21:00	41.4336	4.3812	37.0525	232.6799	3.1054	229.5745
21:00-22:00	3.7524	3.7508	0.0016	2.957	2.7759	0.1811
22:00-23:00	3.8913	3.8561	0.0352	3.7822	2.8327	0.9496
23:00-24:00	3.8913	3.8561	0.0352	3.7822	2.8327	0.9496

In Table 6, the difference between the current and optimal values for the manual lane-type is shown. The outliers in the current and optimal values are removed by the z-score criteria. With the optimal speed limit, the travel time of vehicles in the toll road is decreased by 2.217382674 minutes per vehicle on the average. The mean queue length of vehicles decreased by 14.87835948 vehicles per minute on the average.

For the ACM and ETC lane-types, the comparison of the current and optimal values are shown in Tables 7 and 8. On the average, the travel time of vehicles for ACM lane-type is decreased by 0.007478647 minutes while the queue length is decreased by 0.252180027 vehicles per minute. For the ETC lane-types, there is no decrease in time travel and queue length since the current speed limit is also the optimal speed limit. Thus, it can be deduced that the operation of a toll plaza is at optimality if all tollbooths *are ETC.*

Table 7. Comparison of Current and Optimal Values for the ACM Lane-type

Time	Travel Time			Queue Length		
	Current	Optimal	Decrease	Current	Optimal	Decrease
24:00-1:00	3.1452	3.1452	0	0.1981	0.1981	0
1:00-2:00	3.1471	3.1471	0	0.2112	0.2112	0
2:00-3:00	3.1512	3.1512	0	0.2394	0.2394	0
3:00-4:00	3.1681	3.1681	0	0.3644	0.3644	0
4:00-5:00	3.1603	3.1603	0	0.3052	0.3052	0
5:00-6:00	3.2676	3.2676	0	1.2382	1.2382	0
6:00-7:00	3.5046	3.5046	0	3.5578	3.5578	0
7:00-8:00	3.4055	3.4055	0	2.5727	2.5727	0
8:00-9:00	3.241	3.241	0	0.9915	0.9915	0
9:00-10:00	3.6555	3.6238	0.0316	5.0725	3.7384	1.334
10:00-11:00	3.3002	3.3002	0	1.5464	1.5464	0
11:00-12:00	3.2299	3.2299	0	0.8899	0.8899	0
12:00-13:00	3.2069	3.2069	0	0.6857	0.6857	0
13:00-14:00	3.219	3.219	0	0.792	0.792	0
14:00-15:00	3.2598	3.2598	0	1.1643	1.1643	0
15:00-16:00	3.3659	3.3659	0	2.1832	2.1832	0
16:00-17:00	3.5397	3.5386	0.0012	3.9091	3.6796	0.2295
17:00-18:00	3.7541	3.6749	0.0792	6.0686	3.7733	2.2953
18:00-19:00	3.2368	3.2368	0	0.9531	0.9531	0
19:00-20:00	3.947	3.743	0.2041	8.0229	3.8194	4.2035
20:00-21:00	3.2761	3.2761	0	1.3178	1.3178	0
21:00-22:00	3.7179	3.6579	0.06	5.703	3.7617	1.9413
22:00-23:00	3.2069	3.2069	0	0.6857	0.6857	0
23:00-24:00	3.1628	3.1628	0	0.3239	0.3239	0

Table 8. Comparison of Current and Optimal Values for the ETC Lane-type

Time	Travel Time			Queue Length		
	Current	Optimal	Decrease	Current	Optimal	Decrease
24:00-1:00	3.0209	3.0209	0	0.0026	0.0026	0
1:00-2:00	3.021	3.021	0	0.0033	0.0033	0
2:00-3:00	3.021	3.021	0	0.0029	0.0029	0
3:00-4:00	3.0212	3.0212	0	0.0046	0.0046	0
4:00-5:00	3.0217	3.0217	0	0.0092	0.0092	0
5:00-6:00	3.0224	3.0224	0	0.0187	0.0187	0
6:00-7:00	3.0297	3.0297	0	0.2073	0.2073	0
7:00-8:00	3.0397	3.0397	0	0.5969	0.5969	0
8:00-9:00	3.0319	3.0319	0	0.2866	0.2866	0
9:00-10:00	3.0273	3.0273	0	0.1313	0.1313	0
10:00-11:00	3.0254	3.0254	0	0.0784	0.0784	0
11:00-12:00	3.0242	3.0242	0	0.0515	0.0515	0
12:00-13:00	3.0238	3.0238	0	0.0433	0.0433	0
13:00-14:00	3.0241	3.0241	0	0.0477	0.0477	0
14:00-15:00	3.0246	3.0246	0	0.0597	0.0597	0
15:00-16:00	3.0257	3.0257	0	0.0864	0.0864	0
16:00-17:00	3.0267	3.0267	0	0.1137	0.1137	0
17:00-18:00	3.0283	3.0283	0	0.1616	0.1616	0
18:00-19:00	3.0318	3.0318	0	0.2832	0.2832	0
19:00-20:00	3.0311	3.0311	0	0.2567	0.2567	0
20:00-21:00	3.0287	3.0287	0	0.175	0.175	0
21:00-22:00	3.0236	3.0236	0	0.0383	0.0383	0
22:00-23:00	3.022	3.022	0	0.0126	0.0126	0
23:00-24:00	3.0211	3.0211	0	0.0042	0.0042	0

In our experiment in the one lane-type case, results shows that the speed limit significantly affects the queue length and travel time of vehicles on the toll road.

6. Conclusion

Queue length for payment and travel time are two factors looked upon by drivers in assessing the performance of a toll road system. With a great volume of vehicles plying a toll road system, the queue length of vehicles waiting for payment in the toll plaza increases, as well as the travel time. This research investigated the effects of an imposed speed limit on the queue length of vehicles and on travel time. Moreover, it aimed to identify the optimal speed limit that would improve the performance of the toll road system by maintaining the queue length within an arbitrary value and minimizing the travel time of vehicles.

A nonlinear program model (P) was developed for a toll road with a toll plaza operating with only one lane-type or tollbooth which aims to minimize the travel time and maintains the queue length within an arbitrarily set value. The nonlinear program was simplified, and formula (Theorem 4.2) was developed that identifies the optimal speed limit.

The formula for the one lane-type case model was numerically tested using the values in [4]. Results show that the optimal speed limit obtained by the formula maintained the queue length of vehicles within the arbitrarily set value. A significant decrease in the mean queue length of vehicles can be observed in comparing the values with the optimal and current speed limits. The observation is true for the three lane-types given. The travel time of vehicles also improved in the optimal speed limit compared to the use of the current speed limit.

Future studies may be conducted on toll plazas operating with various lane types, or on toll roads with multiple entrance and exit points. To enhance the results of this study, a survey on driver-accepted queue lengths may be done to get appropriate values of m. Lastly, other factors affecting travel time can be taken into consideration, e.g., road mishaps or accidents, road conditions, and vehicle fuel consumption.

References

Busam, A. K. (2005). Optimization of waiting time at toll plazas. FIU Electronic Theses and Dissertations, 1898. https://digitalcommons.fiu.edu/etd/1898
Edie, L. C. (1954). Traffic Delays at Toll Booths. Journal of the operations research society of America, 2(2), 107-138. https://doi.org/10.1287/opre.2.2.107
Kesten, A. S., Göksu, G., & Akbaş, A. (2013). Dynamic ramp metering approach for an urban highway using microscopic traffic simulation. In 2013 13th International Conference on ITS Telecommunications (ITST) (pp. 90-96). IEEE. https://doi.org/10.1109/ITST.2013.6685527
Kim, S. (2009). The toll plaza optimization problem: Design, operations, and strategies. Transportation Research Part E: Logistics and Transportation Review, 45(1), 125-137. https://doi.org/10.1016/j.tre.2008.03.004
Kumar, A., Thakare, A., & Tawalare, A. (2020). Strategy to Reduce Queuing Time at Toll Plaza. In Transportation Research: Proceedings of CTRG 2017 (pp. 453-464). Springer. https://doi.org/10.1007/978-981-32-9042-6_36
Mittal, H., & Sharma, N. (2022). Operational Optimization of Toll Plaza Queue Length Using Microscopic Simulation VISSIM Model. Journal of Algebraic Statistics, 13(1), 418-425. https://doi.org/10.52783/jas.v13i1.100
Nuredini, R., & Ramadani, J. (2011). Performance Measurment Model of Pay-Toll System. In Proceedings of the Eighth Conference On Informatics And Information Technology (pp. 77-79). Faculty of Computer Science and Engineering: Ss Cyril and Methodius University - Skopje, Macedonia. http://ciit.finki.ukim.mk/data/papers/8CiiT/8CiiT-16.pdf
O'Dea, W. P. (1999). Optimal metering in the bottleneck congestion model. Journal of Urban Economics, 45(1), 125-142. https://doi.org/10.1006/juec.1998.2085
Roess, R. P., Prassas, E. S., & McShane, W. R. (2004). Traffic Engineering (3rd ed.). Prentice Hall. https://www.pearsonhighered.com/assets/preface/0/1/3/4/0134599713.pdf
Van Dijk, N. M., Hermans, M. D., Teunisse, M. J., & Schuurman, H. (1999). Designing the Westercheldetunnel toll plaza using a combination of queueing and simulation. In Proceedings of the 31st conference on Winter simulation: Simulation---a bridge to the future-Volume 2 (pp. 1272-1279). IEEE. https://doi.org/10.1109/WSC.1999.816853
Wolshon, B., & Lambert, L. (2004). Convertible Roadways and Lanes. Washington, DC: The National Academies Press. https://doi.org/10.17226/23331
Yang, X., Lin, Y., Lu, Y., & Zou, N. (2013). Optimal variable speed limit control for real-time freeway congestions. Procedia-Social and Behavioral Sciences, 96, 2362-2372. https://doi.org/10.1016/j.sbspro.2013.08.265